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ON BOUNDARY INTEGRAL EQUATIONS IN ELECTROELASTICITY*

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A class of plane electroelasticity problems of the steady vibrations of bodies with a smooth boundary is studied. A system of boundary integral equations for the components of the displacement vector and the potential is formulated on the basis of the fundamental solution constructed and its analysis.

1. Let the body occupy a two-dimensional connected domain Ω in the x_1x_3, plane bounded by a smooth closed contour L. Let $L = L^1 \cup L^2$ $(L^1 \cap L^2 = \varnothing)$, where the part L^1 of the boundary is electroded while the other part L^2 is not.

We apply a generalization of the Betti reciprocity theorem to the case of an electroelastic medium /1/ to derive the fundamental system of integral equations by assuming that the vibrations mode is steady and obeys the law $\exp(-i\omega t)$. In conformity with the theorem, we examine two states of the medium $u_1^{(n)}, u_3^{(n)}, \varphi^{(n)}, \sigma_{ij}^{(n)}, D_k^{(n)}, \quad n = 1, 2$.

These states are described by a system of electroelasticity equations (here, unlike /1/, we consider an inhomogeneous equation for the electric field)

$$\sigma_{ij,j}^{(n)} + X_i^{(n)} + \rho\omega^2 u_i^{(n)} = 0, \quad i = 1,3; \quad D_{k,k}^{(n)} + f^{(n)} = 0; \quad n = 1,2 \tag{1.1}$$

where f is the electric charge density, and D_k are the components of the electric induction vector. Taking account of the governing relationships $\sigma_{ij}^{(n)} = c_{ijkl}\varepsilon_{kl}^{(n)} - e_{kij}E_k^{(n)}, \quad n = 1, 2$ we have from the first two equations in (1.1) /1/

$$\int_\Omega (X_i^{(1)}u_i^{(2)} - X_i^{(2)}u_i^{(1)})\, d\Omega + \int_L (p_i^{(1)}u_i^{(2)} - p_i^{(2)}u_i^{(1)})\, dL = e_{kij}\int_\Omega (\varepsilon_{ij}^{(1)}E_k^{(2)} - \varepsilon_{ij}^{(1)}E_k^{(2)})\, d\Omega \tag{1.2}$$

Using the relationship

$$D_k^{(n)} = e_{kij}\varepsilon_{ij}^{(n)} + \varepsilon_{kj}E_j^{(n)}, \quad n = 1, 2$$

and considering the electric field potential: $E_k = -\varphi_{,k}$, we analogously obtain

$$-\int_L (D_k^{(1)}\varphi^{(2)} - D_k^{(2)}\varphi^{(1)})\, n_k\, dL - \int_\Omega (f^{(1)}\varphi^{(2)} - f^{(2)}\varphi^{(1)})\, d\Omega = e_{kij}\int_\Omega (\varepsilon_{ij}^{(1)}E_k^{(2)} - \varepsilon_{ij}^{(2)}E_k^{(1)})\, d\Omega \tag{1.3}$$

from the last relationship in (1.1).

Comparing (1.2) and (1.3), we have

$$\int_\Omega F_i^{(1)}\chi_i^{(2)}\, d\Omega + \int_L T_{ij}^{(1)}\chi_j^{(2)}n_i\, dL = \int_\Omega F_i^{(2)}\chi_i^{(1)}\, d\Omega + \int_L T_{ij}^{(2)}\chi_j^{(1)}n_i dL \tag{1.4}$$

$$\{u_1, \varphi, u_3\} = \{\chi_1, \chi_2, \chi_3\}, \quad \{X_1, f, X_3\} = \{F_1, F_2, F_3\}$$

$$T_{ij} = \sigma_{ij}, \quad T_{i2} = D_i, \quad i, j = 1, 3$$

Considering the desired displacement distribution as the first state in (1.4) and the potential for $F_i^{(1)} = 0, \chi_j^{(1)} = \chi_j, T_{ij}^{(1)} = T_{ij}$, as the second (known), we select the state corresponding to a concentrated generalized load at the point $x = \xi$, where $x = (x_1, x_3)$, $\xi = (\xi_1, \xi_3)$: $F_i^{(2)} = \delta_{im}\delta(x, \xi)$ (δ_{im} is the Kronecker delta, and $\delta(x, \xi)$ is a two-dimensional delta-function). The fundamental solution $\Psi_j^{(m)}$ of system (1.1) corresponds to this generalized load. We therefore obtain from (1.4)

$$\int_L T_{ij}(x_1, x_3)\, \Psi_j^{(m)}(\xi_1 - x_1, \xi_3 - x_3)\, n_i(x_1, x_3)\, dL_x = \tag{1.5}$$

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$$\chi_m(\xi_1, \xi_3) + \int_L A^{(m)}(\xi_1, \xi_3, x_1, x_3)\, dL_x, \quad \xi \in \Omega$$

$$A^{(m)}(\xi_1, \xi_3, x_1, x_3) = T_{ij}^{(m)}(\xi_1 - x_1, \xi_3 - x_3)\,\chi_j(x_1, x_3)\, n_i(x_1, x_3).$$

Relationship (1.5) enables us to find the displacement and the potential χ_j within the body if they are known on its boundary.

2. Let us construct the fundamental solution of system (1.1) of electroelasticity equations in the case of practical importance when the material of the medium is a piezoceramic, polarized in the direction of the x_3-axis.

Solving system (1.1) in this case /2/ by using a two-dimensional Fourier integral representation, we have

$$\Psi_k^{(m)}(t_1, t_2) = \frac{1}{4\pi^2} \iint_\Gamma P_k^{(m)}(\alpha_1, \alpha_3, k) \exp[i(\alpha_1 t_1 + \alpha_3 t_3)]\, d\alpha_1\, d\alpha_3 \tag{2.1}$$

$$P_k^{(m)}(\alpha_1, \alpha_3, k) = \frac{P_{km}(\alpha_1, \alpha_3, k)}{p_0(\alpha_1, \alpha_3, k)}, \quad k = \omega \sqrt{\frac{\rho}{c_{33}}}$$

$$p_0(\alpha_1, \alpha_3, k) = \begin{vmatrix} c_{11}\alpha_1^2 + c_{44}\alpha_3^2 - c_{33}k^2 & (e_{15} + e_{31})\alpha_1\alpha_3 & (c_{13} + c_{44})\alpha_1\alpha_3 \\ (e_{15} + e_{31})\alpha_1\alpha_3 & -\mathfrak{e}_{11}\alpha_1^2 - \mathfrak{e}_{33}\alpha_3^2 & e_{15}\alpha_1^2 + e_{33}\alpha_3^2 \\ (c_{13} + c_{44})\alpha_1\alpha_3 & e_{15}\alpha_1^2 + e_{33}\alpha_3^2 & c_{44}\alpha_1^2 + c_{33}\alpha_3^2 - c_{33}k^2 \end{vmatrix}$$

Here Γ is a surface coinciding everywhere with the plane R^2 with the exception of a set of real zeros of the polynomial $p_0(\alpha_1, \alpha_3, k)$, which it envelopes in confirmity with the principle of ultimate absorption /3/ $p_{km}(\alpha_1, \alpha_3, k)$ is obtained from $p_0(\alpha_1, \alpha_3, k)$ by replacing the k-th column by the column $(\delta_{1m}, \delta_{2m}, \delta_{3m})$.

Let us investigate the structure of the set of zeros $p_0(\alpha_1, \alpha_3, k)$. Changing to dimensionless coordinates $\alpha_1 = k\beta\cos\psi$, $\alpha_3 = k\beta\sin\psi$ and taking into account the homogeneity of the polynomial $p_0(\alpha_1, \alpha_3, k)$, we obtain

$$p_0(\alpha_1, \alpha_3, k) = k^6 p_0(\beta\cos\psi, \beta\sin\psi, 1) = k^6 F_{01}\beta^2[\beta^2 - R_1^2][\beta^2 - R_2^2] = \beta^2(F_{01}\beta^4 + F_{02}\beta^2 + F_{03})$$

$$p_{km}(\alpha_1, \alpha_3, k) = k^4 p_{km}(\beta\cos\psi, \beta\sin\psi, 1) = F_{k1}^{(m)}\beta^4 + F_{k2}^{(m)}\beta^2 + F_{k3}^{(m)}$$

$$F_{01} = p_0(\cos\psi, \sin\psi, 0), \quad F_{k1}^{(m)} = p_{km}(\cos\psi, \sin\psi, 0)$$

$$F_{02} = -c_{33}\left(\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} + \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}\right), \quad F_{03} = c_{33}^2 f_{22}$$

$$F_{12}^{(m)} = -c_{33}\begin{vmatrix} \delta_{1m} & f_{12} \\ \delta_{2m} & f_{22} \end{vmatrix}, \quad F_{22}^{(m)} = -c_{33}\left(\begin{vmatrix} f_{11} & \delta_{1m} \\ f_{12} & \delta_{2m} \end{vmatrix} + \begin{vmatrix} \delta_{2m} & g_{12} \\ \delta_{3m} & g_{22} \end{vmatrix}\right)$$

$$F_{32}^{(m)} = -c_{33}\begin{vmatrix} g_{11} & \delta_{2m} \\ g_{12} & \delta_{3m} \end{vmatrix}, \quad F_{k3}^{(m)} = c_{33}^2\delta_{2k}\delta_{2m}; \quad k = 1, 2, 3$$

$$f_{11} = c_{11}\cos^2\psi + c_{44}\sin^2\psi, \quad f_{12} = (e_{15} + e_{31})\sin\psi\cos\psi$$

$$f_{22} = g_{11} = -(\mathfrak{e}_{11}\cos^2\psi + \mathfrak{e}_{33}\sin^2\psi), \quad g_{12} = e_{15}\cos^2\psi + e_{33}\sin^2\psi$$

$$g_{22} = c_{44}\cos^2\psi + c_{33}\sin^2\psi; \quad R_{1,2}^2 = -(2F_{01})^{-1}[F_{02} \pm (F_{02}^2 - 4F_{01}F_{03})^{1/2}]$$

where $F_{01}(\psi) \neq 0$ for all $\psi \in [0, 2\pi]$. We note that

$$R_k(\psi + \pi) = R_k(\psi) = R_k(-\psi), \quad k = 1, 2$$

Graphs of the functions $R_1(\psi)$ and $R_2(\psi)$ for CdS (the solid lines) and the ceramic TsTS-19 (the dashed lines) are shown in the figure for $\psi \in [0, \pi/2]$ (curves *1* and *2*, respectively). Therefore, the surface Γ in (2.1) ca be represented in the form $\Gamma = \sigma_+(\psi) \times [0, 2\pi]$, where the contour $\sigma_+(\psi)$ issues from the origin and coincides with the real positive semi-axis and derivatives from it at the real poles $R_k(\psi)$ $(k = 1, 2)$ in the lower half-plane.

We further convert (2.1) to the form

$$\Psi_k^{(m)}(r\cos\eta, r\sin\eta) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_{\sigma_+(\psi)} P_k^{(m)}(\beta\cos\psi, \beta\sin\psi, 1)\exp[ikr\cos(\psi - \eta)]\,\beta d\beta d\psi \tag{2.2}$$

$$r = [(\xi_1 - x_1)^2 + (\xi_3 - x_3)^2]^{1/2}, \quad \cos\eta = (\xi_1 - x_1)/r, \quad \sin\eta = (\xi_3 - x_3)/r$$

Using the expansions

$$P_k^{(m)}(\beta\cos\psi, \beta\sin\psi, 1) = \frac{H_{k0}^{(m)}}{\beta^2} + \frac{H_{k1}^{(m)}}{\beta^2 - R_1^2} + \frac{H_{k2}^{(m)}}{\beta^2 - R_2^2}$$

$$H_{kl}^{(m)}(\psi) = \frac{F_{k1}^{(m)} R_l^4 + F_{k2}^{(m)} R_l^2 + F_{k3}^{(m)}}{F_{01} R_l^2 [R_1^2 - R_2^2]}$$

$$H_{k0}^{(m)}(\psi) = \frac{F_{k3}^{(m)}}{F_{01} R_1^2 R_2^2}; \quad k = 1, 2, 3; \quad l = 1, 2$$

we convert equality (2.2) into a form containing just single integrals, as is important for applications

$$\Psi_k^{(m)}(r\cos\eta, r\sin\eta) = \frac{1}{2\pi^2} J_k + \frac{1}{4\pi^2} \int_0^\pi \sum_{j=1}^{2} H_{kj}^{(m)}(\psi)\{\pi i \exp[iz_j(r, \psi, \eta)] -$$

$$2S_j(r, \psi, \eta)\}\, d\psi, \quad k = 1, 2, 3; \quad J_1 = J_3 = 0,$$

$$z_j(r, \psi, \eta) = krR_j(\psi)\,|\cos(\psi - \eta)|$$

$$S_j(r, \psi, \eta) = \cos z_j \operatorname{ci} z_j + \sin z \operatorname{si} z_j; \quad \operatorname{si}(t) = -\int_t^\infty \frac{\sin t}{t}\, dt; \quad \operatorname{ci}(t) = -\int_t^\infty \frac{\cos t}{t}\, dt$$

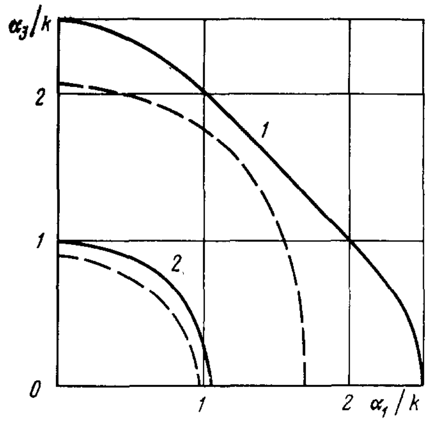
Regularization of the integral J_2 must be realized in calculating $\Psi_2^{(m)}$ by using the concept of the Hadamard finite value /5/

$$J_2 = \int_0^\pi \int_1^\infty H_{20}(\psi) \cos[kr\cos(\psi - \eta)] \frac{d\beta}{\beta}\, d\psi +$$

$$\int_0^\pi \int_0^1 H_{20}(\psi)\{\cos[kr\beta\cos(\psi - \eta)] - 1\} \frac{d\beta}{\beta}\, d\psi =$$

$$\int_0^\pi H_{20}(\psi)[C + \ln|kr\cos(\psi - \eta)|]\, d\psi.$$

where C is Euler's constant.



Thus (2.3) determine the fundamental solution $\Psi_k^{(m)}$ $(k = 1, 2, 3)$ of system (1.1).

3. We will derive the fundamental system of boundary integral equations. To do this the passage to the limit as $\xi \to y \in L$ must be made in (1.5). Applying the well-known procedure /4/, we have (L_ε is an arc of a circle of radius ε with centre at y)

$$\lim_{\xi \to y} \int_L A^{(m)}(\xi_1, \xi_3, x_1, x_3)\, dL_x = \chi_j(y_1, y_3) C_j^{(m)} + \lim_{\varepsilon \to 0} \int_{L - L_\varepsilon} A^{(m)}(y_1, y_3, x_1, x_3)\, dL_x. \tag{3.1}$$

$$C_j^{(m)} = \lim_{\varepsilon \to 0} \int_{L_\varepsilon} T_{ij}^{(m)}(y_1 - x_1, y_3 - x_3)\, n_i(x_1, x_3)\, dL_x$$

We consider the limit of the first integral on the right-hand side of (3.1) by setting $\beta = v/\varepsilon$ in representation (2.2) and going over to the local coordinate system $x_1 = y_1 + \varepsilon\cos\theta$ $x_3 = y_3 + \varepsilon\sin\theta$

$$C_j^{(m)} = -\frac{ik}{4\pi^2} \lim_{\varepsilon \to 0} \Big[\sum_{k=1,2} c_{ijkl} \int_0^{2\pi} N_l(\eta)\, I_{ik}^{(m)}(\varepsilon, \eta)\, d\eta + \tag{3.2}$$

$$e_{lij}\int_0^{2\pi} N_l(\eta)\, I_{i2}^{(m)}(\varepsilon,\eta)\, d\eta\Big], \quad j = 1, 3$$

For $j = 2$ we obtain an analogous expression by replacing e_{ijkl} by e_{ikl} and e_{lij} by ε_{il} in (3.2). Here

$$N_l(\theta) = \begin{cases}\cos\theta, & l = 1,\\ \sin\theta, & l = 3,\end{cases} \quad I_{ik}^{(m)}(\varepsilon,\eta) =$$

$$\int_0^{\pi}\int_{\sigma_+} P_k^{(m)}(v\cos\eta,\ \ v\sin\eta, \varepsilon^2)\, v^2 \exp[-ikv\cos(\eta-\theta)]\, N_i(\theta)\, dv\, d\theta$$

We evaluate the limit

$$\lim_{\varepsilon\to 0} I_{ik}^{(m)}(\varepsilon,\eta) = P_k^{(m)}(\cos\eta, \sin\eta, 0)\int_0^{\pi}\int_0^{\infty} N_i(\theta)\exp[-ikv\cos(\eta-\theta)]\, dv\, d\theta = \tag{3.3}$$

$$-\frac{\pi i}{ik} P_k^{(m)}(\cos\eta, \sin\eta, 0)\left[N_i(\eta) + \frac{4i}{\pi}\sum_{t=1}^{\infty}(-1)^t \frac{t}{4t^2-1} N_i\left(\frac{\pi}{2} - 2t\eta\right)\right]$$

Substituting (3.3) into (3.2) and carrying out the requisite reduction, we find that

$$C_j^{(m)} = -{}^1/_2\delta_{jm} \tag{3.4}$$

The limit of the second integral on the right-hand side of (3.1) is the Cauchy principal value. In passing, we note that

$$\lim_{\varepsilon\to 0}\int_{L_\varepsilon} T_{ij}(x_1, x_3)\, \Psi_j^{(m)}(y_1 - x_1, y_3 - x_3)\, n_i(x_1, x_3)\, dL_x = 0 \tag{3.5}$$

is calculated analogously.

Therefore, to take account of (3.1), (3.4), and (3.5), we obtain after passing to the limit as $\xi \to y$ in (1.5)

$$\int_L T_{ij}(x_1, x_3)\, \Psi_j^{(m)}(y_1 - x_1, y_3 - x_3)\, n_i(x_1, x_3)\, dL_x = \frac{1}{2}\chi_m(y_1, y_3) +$$

$$\text{v. p.}\int_L T_{ij}^{(m)}(y_1 - x_1, y_3 - x_3)\, \chi_j(x_1, x_3)\, n_i(x_1, x_3)\, dL_x, \quad (y_1, y_3) \in L, \quad m = 1, 2, 3$$

Thus, the plane problem of the steady vibrations of an electroelastic medium is reduced to a system of three singular integral equations. We note that the well-developed methods of boundary elements /4/ are sufficiently efficient for systems of this kind, and enable the mechanical and electric fields to be calculated for a broad class of linear electroelasticity problems.

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